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Do natural disasters make sustainable growth impossible?

by

Lee Endress, James Roumasset, and Christopher Wada

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UNIVERSITY OF HAWAIʻI AT MANOA
2424 MAILE WAY, ROOM 540 • HONOLULU, HAWAIʻI 96822
WWW.UHERO.HAWAII.EDU

1 Introduction

The prospect of sustainable growth is largely motivated by the notion of "stewardship for the future." In discussing what the present owes the future, Robert Solow (1991) remarks, *"...what we are obligated to leave behind is a generalized capacity to create well-being, not any particular thing or any particular natural resource."* It is the economy's productive base, as represented by the totality of capital in its various forms, that provides the capacity for creating well-being now and in the future. Arrow et al. (2004) suggest that concerns for sustainability are captured by the *sustainability criterion* – that intertemporal social welfare not decrease over time – and present a mathematical proof that this requires a non-declining value of total capital, including natural and human capital.

The sustainable growth literature provides rules for resource use and investment in the context of perfect certainty. For example, Heal (1998, 2001) and Endress et al. (2005) provide conditions such that optimal resource use and investment result in sustainable consumption, utility, and wealth. In such cases, optimality implies sustainability but not the other way around. That is, maximizing intertemporal welfare satisfies the sustainability criterion. There is no need to specify a sustainability constraint.[1] However, the sustainable growth literature typically begs the question of how rules for optimal resource extraction and investment change under uncertainty, particularly when stocks of physical and/or natural capital are subject to negative shocks. Is optimality now at odds with sustainability? Is sustainable growth even possible?

The traditional literature on stochastic economic growth (as surveyed e.g. by Gollier 2001) emphasizes optimal savings in the face of uncertain income. However, this strand of

[1] In other cases discussed in the same literature, sustaining a positive, non-declining level of consumption is impossible, i.e. optimization is futile.

literature does not adequately capture how natural disasters may affect physical and ecological capital stocks. Recent work on the economics of natural disasters has made significant progress in this direction (e.g., Tsur and Zemel 2006, Barro 2006, Noy 2009, Gollier and Weitzman 2010, Hallegatte 2012, Hallegate and Ghil 2008, Pindyck and Wang 2013, and Weitzman 2012; an earlier contribution is Cropper 1976). Yet, the link between disaster economics and the economics of sustainable growth remains incomplete.

The objective of this paper is to derive insights regarding the implications of disaster risk for the prospects of long-run sustainable growth. In the interest of minimizing model complexity while maximizing interpretability of results, we begin by deriving optimality conditions for a two-period model (Gollier, 2013), adapted for the case of natural disaster. We then show that those conditions remain valid in an infinite-horizon continuous-time setting—with adjustments to account for continuous compounding—provided that the probability of disaster is exogenously and parametrically specified. This result is key to answering the earlier posed question of whether or not the generalized sustainability criterion can be satisfied without constraints in a world where capital stocks are subject to large and uncertain negative shocks. We find that in short of a catastrophe that depresses natural capital below a tipping point, a disaster functions as a restart, and the economy can still approach a golden rule steady state if optimal paths are followed.

2 Two-period model: a stepping stone

A two-period model, constructed and applied to discounting in the presence of uncertainty by Gollier (2013) provides a point of departure for our analysis. Utility (U) is a function of consumption (C) at time 0 and *t*, where *t* is arbitrary. The two periods are not adjacent; they are points in time separated by time span *t*. There is a capital market with interest rate r, and the return to capital is compounded continuously between time 0 and time t. The pure rate of time preference is given by ρ. In this expected utility framework, the Ramsey rule is

(1) $r_t = \rho - (1/t)\log\{EU'(C_t)/U'(C_0)\}$

where E is the expectation operator. To allow for the possibility of a natural disaster, we introduce uncertainty regarding destruction of capital in section 2.1.[2]

2.1 Investment smooths consumption in a two-period model with disaster risk

The Gollier (2013) model extends the Ramsey rule to the case of uncertainty in the growth of consumption. We adapt the Gollier two-period model to the case of natural disaster: there is a probability P that a natural disaster destroys a fraction D of the capital stock K in period t (see the Appendix for a detailed description of the model). Since the uncertain capital stock is an input to production, consumption in the second period is state dependent. The planner's objective is to maximize expected utility, subject to feasibility constraints in each of the two periods.

With comparative static analysis, it can be shown that investment will decrease, remain unchanged or increase in response to an increase in the probability P of disaster, depending on the value of η, the coefficient of relative risk aversion to intertemporal inequality of consumption (see the Appendix). For the commonly considered case of $\eta > 1$ (more risk averse), this means that risk management entails increasing the capital stock even though said capital is exposed to a greater risk. In summary, the two-period model tells us that precautionary investment helps to smooth consumption but is unable to address long-run questions about sustainability.

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[2] Uncertainty in the Gollier model regards the growth rate of output and consumption. Of note, Weitzman (2012) investigates the Ramsey discounting formula in a stochastic context using the Gollier two-period framework.

2.2 *Extended Ramsey condition in the two-period model*

Using results derived in the Appendix and discussed in section 2.1, we construct an extended Ramsey condition that turns out to be a key piece in our sustainability puzzle. We begin with the classic Ramsey condition in the absence of risk:

(2) $r(t) = \rho + \eta g(t)$

where g(t) is the growth rate of consumption at time t.

To extend equation (2) for the case of natural disaster that, with probability P, destroys a fraction D of capital stock K in period t, we proceed as follows. Substituting equation (A4) from the Appendix into equation (1), applying continuous utility discounting with $\beta = e^{-\rho t}$, using the approximation $\log(x + 1) \cong x$, *and* recognizing a first period constraint on capital investment, leads to the following expression:

(3) $r_t = \rho + (1/t)\eta\log\{[A(K_t)^{\alpha}]/[A(K_0)^{\alpha} - K_t]\} - (1/t)\{P[(1 - D)^{\alpha(1-\eta)} - 1]\}$

The first term on the right hand side of (3) is the "impatience effect" identified by Gollier (2013). The second term is the "wealth effect", and the third term the "precautionary effect." For example, a higher pure rate of time preference increases the threshold level that must be achieved and thereby lowers the optimal level of investment. Similarly, a high coefficient of relative risk aversion and/or high growth rate of capital reduce optimal investment via the wealth effect. The third term has the opposite effect on investment. If the precautionary effect is larger, e.g. due to a higher probability of natural disaster, optimal investment is higher.

3 Continuous-time model: is optimal still sustainable?

In order to integrate the concerns of both sustainable growth and resilience, we need a model of capital accumulation (combined with stewardship of ecological resources) in a model of intertemporal welfare in the face of disaster. In this section, we develop a continuous-time analog to equation (3) and show that the steady state, if it exists, depends on the underlying fundamentals of the economy and satisfies a general criterion for sustainability.

3.1 Exogenous and parametrically specified continuous probability distribution of disaster

We motivate the transition to a continuous time model with a continuous probability distribution of disaster. Suppose that P is the probability of a one-time disaster of severity D, where the time of occurrence is distributed uniformly over the interval $[0, \tau]$. The time line $[0, \infty)$ can then be divided into sequential intervals of length τ: $[0, \tau]$, $(\tau, 2\tau]$, … $((N\text{-}1)\tau, N\tau]$, and so forth. For $1 \leq n \leq N$, the probability of disaster occurring in some interval $((n\text{-}1)\tau, n\tau]$ is $\{P + P(1\text{-}P) + P(1 - P)^2 + \ldots + P(1 - P)(N\text{-}1)\} = \{1 - (1 - P)N\}$, which approaches 1 as $N \rightarrow \infty$. The continuous-time analogue to this distribution is obtained by dividing every interval of span τ (from time 0 to $N\tau$) into m smaller subintervals. As $m \rightarrow \infty$, the probability that an event occurs by time t is $[1 - e^{-Pt}]$. In other words, we have a probability distribution $F(t) = 1 - e^{-Pt}$ with density function $f(t) = Pe^{-Pt}$. P can be interpreted as a hazard rate, analogous to h(Q(t)) in Tsur and Zemel (2006), but constant in our case.

3.2 Extended Ramsey condition and precautionary investment in continuous time

With the shift to the continuous time framework, the term (1/t)P in equation (3) needs to be replaced by the probability density function $f(t) = Pe^{-Pt}$, and the risk of disaster becomes $[Pe^{-Pt}]\,[(1 - D)^{\alpha(1-\eta)} - 1]$. Note that integrating $[Pe^{-Pt}][(1 - D)^{\alpha(1-\eta)} - 1]$ from 0 to time T gives cumulative precautionary investment at time T of $[1 - e^{-PT}][(1 - D)^{\alpha(1-\eta)} - 1]]$, where the first factor approaches 1 as $T \rightarrow \infty$. The extended Ramsey equation can now be written as

$$(4) \qquad r_t = \rho + \eta g - [Pe^{-Pt}][(1 - D)^{\alpha(1-\eta)} - 1]$$

We offer some observations regarding the effect of parameters η and P on precautionary investment $\{- [Pe^{-Pt}][(1 - D)^{\alpha(1-\eta)} - 1]\}$: When $\eta > 1$, the product of the two terms in square brackets is positive; with the negative sign out front, r is lowered, indicating a higher level of capital at time t; that is precautionary investment is positive. For the case $\eta < 1$, we have the opposite story; precautionary investment is now negative. And for $\eta = 1$, there is no precautionary investment. These results are consistent with those presented in section 2.1.

The situation for P is more nuanced than in the discrete two-period model. Take the usual case $\eta > 1$. The second term, $[(1 - D)^{\alpha(1-\eta)} - 1]$, is a constant for fixed parameters α, D, and η. Now suppose hazard rate P increases so that $dP > 0$. Consider the density function as a function of P for a fixed $t = s$: $f(P, s) = Pe^{-Ps}$. The derivative of f with respect to P, $f'(P, s) = [1 - Ps]e^{-Ps}$, is positive when $[1 - Ps] > 0$ or $s < 1/P$. This result implies that an increase in hazard rate P should motivate higher precautionary investment earlier, and lower investment later. This makes intuitive sense. Plotting the density function f(t) for two hazard rates, $P_1 < P_2$, illustrates the situation. The density for P_2 starts out higher, but drops off more quickly; at some time t = s, the two curves cross over; as densities, both functions integrate out to probability 1 over an infinite time horizon.

So far in this analysis we have assumed 100% capital depreciation represented by $\delta = 1$. How might a lower rate of capital depreciation, $\delta < 1$, affect precautionary investment? We show in the Appendix that the precautionary investment term in the extended Ramsey equation is adjusted by a factor Z which depends inversely on the ratio of the marginal product of capital (MPK) to the term $(1 - \delta)$: $\{ - [Pe^{-Pt}][(1 - D)^{\alpha(1-\eta)}(Z) - 1]\}$. In earlier times t, when capital accumulation is low, the ratio MPK/ $(1 - \delta)$ is relatively high and the factor Z is close to 1 in value. As MPK/ $(1 - \delta)$ declines over time, Z increases (to about 1.23 as MPK approaches δ in the example presented in the Appendix); however, this increase is more than offset by the decline in weighting by the density function [Pe^{-Pt}] as t increases. We conclude that the factor Z is not significant and neglect it in the subsequent analysis.

3.3 Long run steady state in the absence of risk

How do we specify the long run steady state of an economy following the optimal path? Consider first the Ramsey condition (2) for the case of certainty and no risk as a benchmark for comparison with the case of disaster risk. The steady state, if it exists, is characterized by $g = 0$; the growth rate of consumption is zero, and consumption attains the constant level C^*. As a consequence, equation (2) simplifies to $r^* = \rho$. The steady state discount rate r^* satisfies $r^* = F'(K^*) - \delta$. So in the steady state, $F'(K^*) = (\delta + \rho)$. If we take $\rho = 0$, neutral weighting of utility across time as a pillar of sustainability (Endress et al., 2005), then our steady state result is $F'(K^*) = \delta$. This is the classic "Golden Rule" of capital accumulation in the context of long run sustainability.[3]

Assuming that $0 < \alpha < 1$, the long run steady state can be specified by solving for K^*.[4] For $F(K) = AK^{\alpha}$, $F'(K^*) = \alpha A(K^*)^{\alpha-1} = \delta$, which means that $K^* = \{\delta/(\alpha A)\}^{1/(\alpha-1)} = \{(\alpha A)/(\delta)\}^{1/(1-\alpha)}$. For the case $\delta = 1$ (100% capital depreciation), $K^* = (\alpha A)^{1/(1-\alpha)}$. While K^* depends on the technology of production as characterized by α (output elasticity of capital) and A (total factor productivity), it does not depend on the initial stock of capital K_0. We can then solve for the golden rule level of steady state consumption, $C^* = F(K^*) - \delta K^*$. Its associated utility, $U(C^*)$, is an important component in the formulation of social wellbeing across time, usually termed

[3] For $\rho > 0$, the steady state result is usually called the "Modified Golden Rule."

[4] Otherwise $K^* = (\alpha A)^{1/(1-\alpha)}$ has no solution. When $\alpha = 1$, $F'(K) = A$, which in the general case will not equal δ. We could, however, take $\alpha = (1 - \varepsilon)$ and explore the limit as ε goes to zero.

intertemporal welfare. One measure of wellbeing at a particular point in time t (instantaneous wellbeing), is the gap between $U[C(t)]$ and $U[C^*]$. Now for $t < \infty$, $\{U[C(t)] - U[C^*]\} < 0$. As the economy grows and both $C(t)$ and $U[C(t)]$ increase, the gap narrows, becoming less negative.

A more illuminating extension is the summing up of instantaneous wellbeing of society over time, starting at time t, to yield intertemporal welfare, denoted $V(t)$. Since the model is formulated in continuous time, the summing up is achieved through the process of integration. Accordingly, we can write

(5) $$V(t) = \int_{s=t}^{\infty} \{U[C(s)] - U[C^*]\} ds$$

Since the integrand is negative for $0 \leq s < \infty$, $V(t) < 0$. This representation is an integral transformation, due to Koopmans (1965), that yields a finite value of the integral over an infinite time horizon without discounting (i.e., $\rho = 0$). See Endress et al. (2005) for further discussion of this technique and its application to sustainable growth.

A goal of economic policy should be to maximize the intertemporal welfare of society starting from initial time $t = 0$, subject to the feasibility constraint of the economy. Letting $V = V(0)$, the problem can now be stated as:

(6) $$Max\,V = Max \int_{s=t_0}^{\infty} \{U[C(s)] - U[C^*]\} ds, \text{ subject to}$$

$$\dot{K} = F[K(t)] - \delta K(t) - C(t) = A[K(t)]^{\alpha} - \delta K(t) - C(t)$$

Solution of this problem leads directly to the first order efficiency condition, the Ramsey rule, equation (2) with $\rho = 0$. Note that in maximizing V, we are choosing a path for $U[C(s)]$ that makes the integral $\int_{t_0}^{\infty} \{U[C(s)] - U[C^*]\} ds$ the least negative. A key consequence of following the optimal path, as guided by the Ramsey rule, is that intertemporal welfare is non-declining over time; that is $\dot{V} \geq 0$. This observation is fully compatible with the result presented in Arrow et al. (2004). The steady state is characterized by $\dot{K} = 0$ in (6) and $g = 0$ in (2). The economy approaches the steady state as t approaches infinity; $U[C(t)]$ approaches $U[C^*]$, and $V(t)$ approaches zero from below. No ad hoc sustainability constraints are required to achieve optimality. In fact, imposing such constraints serves only to reduce intertemporal welfare.

3.4 Paths of convergence to the steady state with and without disaster risk

In this section, we show that the optimal steady state capital stock does not depend on the initial capital stock or the probability of disaster. It is tied only to the fundamental parameters of the economy, which means that any negative shock to K simply acts as a reset; the long-run target remains unchanged.

3.4.1 Path of convergence without risk of disaster

A closed form solution for the path of $K(t)$ from initial time zero to the steady state typically depends on approximation techniques, such as the process of log-linearization applied to a modified form of the Ramsey condition. The process involves taking logs of key variables followed by performing a first order Taylor series expansion around steady state variables. The result is an approximate path of convergence of $\log[K(t)]$ to the steady state value $\log[K^*]$:

(7) $$\log[K(t)] = [1 - e^{-\sigma t}]\log[K^*] + [e^{-\sigma t}]\log[K_0]$$

The parameter σ is known as the speed of convergence; its value is determined by the parameters of preferences (η, ρ) and production (α, δ, but surprisingly not A, which produces opposite, offsetting effects); however, it does not depend on the initial stock of capital, K_0. Empirical estimates of σ range from 0.015 to 0.03 with $\sigma = 0.02$ corresponding to $\alpha = 3/4$.[5]

A concept related to the transition to the steady state that has direct application to the problem of economic resilience and recovery from disaster is the so-called "half-life of

[5] See Barro and Sala-i-Martin (2004) for a detailed discussion of convergence.

convergence" or $T_{1/2}$. The half-life of convergence is the time interval required to close the gap between log[K_0] and log[K^*] by one half. Algebraic manipulation of equation (7) shows that $T_{1/2}$ = [log 2]/σ; for $\sigma = 0.02$, $T_{1/2} = 0.693/0.02 = 34.66$. So with time measured in years, the half-life of convergence is about 35 years; to close the gap again by half is another 35 years, and yet again, a third half-life of 35 years. Adding up: 105 years, amounting to several generations, are required to close the gap between log[K_0] and log[K^*] by 1/2 + 1/4 + 1/8 = 7/8 of the way + 1/4 + 1/8 = 7/8 of the way.

3.4.2 Path of convergence for the case of a single-event disaster

Suppose that a disaster occurs at time τ_1, $0 < \tau_1 < \infty$, destroying fraction D of accumulated capital stock $K(\tau_1)$. Because of precautionary investment, $K(\tau_1)$ for the case of uncertainty and risk, exceeds the level $K(\tau_1)$ that would apply in the case of certainty and no risk of disaster. Capital remaining is $(1 - D)K(\tau_1)$, which functions as the initial capital stock for the trajectory from time τ_1 forward. The post-disaster trajectory of capital is characterized as follows:

(8) $\log[K(t - \tau_1)] = [1 - \exp[-\sigma(t-\tau_1)]\log[K^*] + \exp[-\sigma(t-\tau_1)] \log[(1 – D)K(\tau_1)]$

In spite of disaster occurring at time τ_1, the values for K^*, σ, and $T_{1/2}$ remain unchanged because they are tied to the fundamental parameters of the economy: steady state capital K^* is determined by A, α, δ and ρ; speed of convergence σ depends on α, δ, η and ρ; $T_{1/2}$ = [log 2]/σ. K^*, σ, and $T_{1/2}$ do not depend on initial capital stock nor on the probability of disaster. Equation (8) represents a reset and restart of the economy from time of disaster τ_1 with $[(1 – D)K(\tau_1)]$ in place of K_0 serving as the "initial" capital stock.

With respect to resilience and recovery, we can estimate the time interval $T = (t - \tau_1)$ required to restore the capital stock back to $K(\tau_1)$ from $[(1 - D)K(\tau_1)]$. Substituting log $[K(\tau_1)]$ on the left hand side of (8) and solving for $(t - \tau_1) = T$ yields $T = (1/\sigma)\log\{1 + [\log(1 -$

D)]/[log(K(τ_1)/K*)]}. For example, with $\sigma = 0.02$, $(1 - D) = 3/4$, and $(K(\tau')/K^*) = 1/4$, $T = 9.4$ years. Robust recovery is contingent on the economy's ability to follow the efficient path approximated by equation (8).

Figure 1 depicts the paths of $U[C(t)]$ and $V(t)$ in the disaster scenario. In the upper graph, $U[C(t)]$ rises along the optimal path from $t = 0$ to $t = \tau_1$, at which time a natural disaster occurs. The capital stock drops from $K(\tau_1)$ to $(1 - D)K(\tau_1)$, resulting in a discontinuous drop in consumption, and hence utility of consumption at time τ_1. Recovery takes place over the time interval T, as computed above. After recovery, $U(C)$ continues the asymptotic approach to $U(C^*)$. The lower graph shows the trajectory of intertemporal welfare, $V(t)$. It rises at a healthy rate, until disaster occurs at time τ_1.

However, since $V(t)$ is an integral, it is continuous (though not differentiable) at time τ_1; there is no jump drop.[6] In fact, if $U(C)$ takes a one-time drop and then rises again during recovery, $V(t)$ remains non-decreasing. In the event that $U(C)$ continued to decline for a period during the recovery phase, $\dot{V}$ would be negative over the same period.[7] After recovery, $V(t)$ continues the climb to the long run steady state, asymptotically approaching zero from below.

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[6] See e.g. Friedman (1999), section 4.5, theorem 1, page 127.

[7] Arrow et al. (2004) remark on the possibility of periods where $\dot{V} < 0$, but don't relate that possibility to disaster.

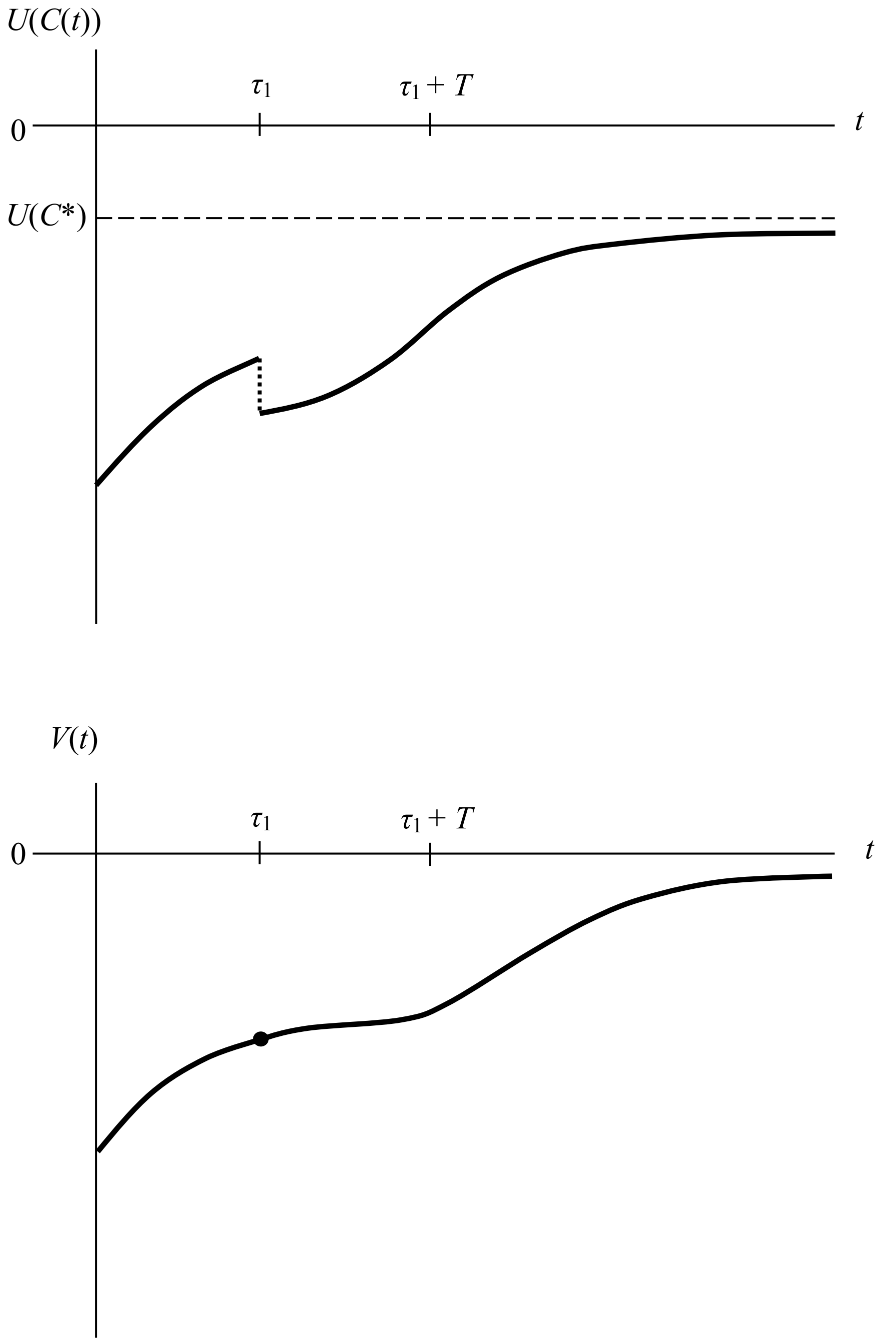

Figure 1. Trajectories of $U(C(t))$ and $V(t)$. Disaster occurs at time τ_1. Duration of recovery is T.

3.4.3 Path of convergence for the case of recurrent disasters

Now suppose we have recurrent disaster events at times τ_1, τ_2, τ_3, ... τ_n, and so forth. After the nth disaster, the extended Ramsey equation is written

(9) $r_t = \rho + \eta g - [P\exp(-P(t - \tau_n))][(1 - D)^{\alpha(1-\eta)} - 1]$,

where we assume that the hazard rate P and severity of disaster D remain constant. The trajectory of the economy from time τ_n forward is now approximated by the equation

(10) $\log[K(t - \tau_n)] = [1 - \exp[-\sigma(t-\tau_n)]\log[K^*] + \exp[-\sigma(t-\tau_n)] \log[(1 – D)K(\tau_n)]$,

which applies until the next disaster event occurs at time $t = \tau_{(n+1)}$. Equations (9) and (10) require modification by substituting $\tau_{(n+1)}$ for τ_n throughout. These modified equations then apply until yet again the next disaster occurs.

Even in the case of certainty and no risk of disaster, the economy approaches, but never reaches, the steady state. The example in section 3.4.1, using representative parameter values, estimates a half-life of convergence to be about 35 years; this represents the time it takes to close the gap between $\log[K_0]$ and $\log[K^*]$ by ½. Three half-lives, or about 105 years, would be required to close the gap 7/8 of the way. We keep climbing the hill, but never quite get to the top.

The climb is much more challenging with the prospect of disaster recurring a countably infinite number of times. This situation might be characterized as a "Stochastic Sisyphus" problem.[8] Each disaster, occurring at some random time, is like the boulder falling down the hill, but not all the way. So in this scenario, Sisyphus climbs higher and higher, but forever suffers partial resets and restarts, never reaching the top. Nonetheless, Sisyphus does get closer and closer: eternally proceeding two miles forward and one mile back, but with the goal of attaining golden rule utility U(C*) at the top of the hill "at the end of time." For this economy, burdened as it is by recurrent disasters, and even for Sisyphus, golden rule utility U(C*), which applies to the case of neutral weighting of utility across time ($\rho = 0$), has special significance. As shown in

[8] In Greek mythology, Sisyphus' punishment for excessive avarice and deceit was to roll a boulder up a hill, only to watch it roll back down, and to repeat this action indefinitely.

Endress et al. (2005) and based on Weitzman (1976), U(C*) represents constant net national product in utility units for all time periods along the optimum trajectory. As figure 1 suggests, the optimum trajectory of utility may be comprised of piecewise continuous segments, with points of discontinuity and downward resets at times of disaster τ_1, τ_2, τ_3, ... τ_n, ...; such is the eternal fate of Sisyphus.

4 Modeling natural resources as a separate sector

Additional insights are generated by incorporating a separate natural resource sector into the model, rather than just including natural resources in total capital K. A general approach is to consider a renewable natural resource that provides ecological services in production as well as amenity value in utility.

4.1 Separate natural resources sector: preliminaries

At time t, a renewable resource is harvested at rate H(t) to serve as a factor of production along with capital: $F(K,H)$. Marginal products are written as F_K and F_H. The remaining stock X at each point in time contributes to amenity value so that utility is a function of both consumption C and natural resource stock X: $U(C,X)$. Marginal utilities are written as U_C and U_X.

Renewal, or regeneration, of the resource is specified by a growth function $G(X)$ that depends on X, the stock level. We follow the renewable resource literature (e.g., Clark, 1990) by basing $G(X)$ on a logistic type growth formula, modified by a factor representing the so-called Allee effect: growth becomes severely depressed at low densities or stock levels, leading to extinction.[9]

[9] The effect is named after American zoologist, Warden Allee. See discussion in Sheffer (2009). Clark (1990) refers to this effect as “critical depensation.”

(11) $G(X) = \pi X[1 - X/(X_{max})][(X - X_{cr})/(X_{max})]$

where π is the intrinsic growth rate of the resource, X_{max} is the maximum stock level, or biological carrying capacity of the natural resource system, and X_{cr} is the critical stock level, at or below which the growth rate turns negative and the resource stock X approaches zero (i.e., extinction). See Figure 2 for a graphical depiction.

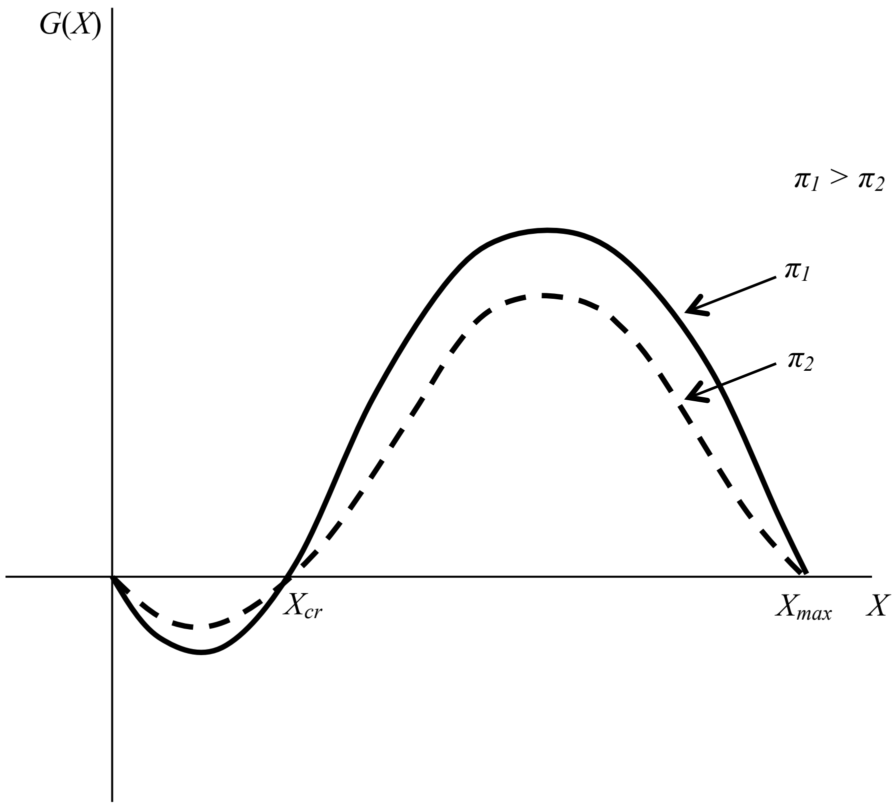

Figure 2. Renewable Resource Growth Function: $G(X) = \pi X[1 - X/X_{max}][(X - X_{cr})/X_{max}]$

The time rate of change in the resource stock level is governed by two key factors, the regeneration rate $G(X)$, and the harvest rate H.

(12) $\dot{X} = G(X) - H$

The natural resource system approaches a long run steady state as time t approaches infinity if $\dot{X}$ approaches zero so that $G(X) = H$. We designate the steady state stock level as X^*.

Unit harvesting cost, $\Phi(X)$ is a function of the stock level and is assumed to be non-decreasing as the stock X is drawn down: $\Phi'(X) \leq 0$.

4.2 *Extended Ramsey and Hotelling rules*

The planner's problem now becomes

(13) $$Max\,V = Max \int_{s=0}^{\infty} \{U[C(s), X(s)] - u[C^*, X^*]\} ds, \text{ subject to}$$

$$\dot{K} = F(K, H) - \delta K - \Phi(X)H - C$$

$$\dot{X} = G(X) - H$$

where initial stock level X_0 satisfies $X_{cr} < X_0 \leq X_{max}$. As in section 3.1, the probability of natural disaster has density function Pe^{-Pt}. Optimization is guided by two first order conditions.

(14) (i) An extended Ramsey rule with $\rho = 0$, as before:

$$r(t) = \eta g - [Pe^{-Pt}][(1 - D)^{\alpha(1-\eta)} - 1]$$

(ii) An extended, renewable resource version of the Hotelling rule (Hotelling, 1931):

$$[F_H - \Phi(X)] = [1/(F_K - \delta)]\{\dot{F}_H + [F_H - \Phi(X)]G'(X) - \Phi'(X)G(X) + U_X/U_C\}$$

The effect of a natural disaster that damages a portion D_X of the resource stock is discussed in section 4.3. For the golden rule steady state, these conditions reduce to

(15) (i′) $r = F_K - \delta = 0$, and

(ii′) $F_H = \Phi(X^*) + \{\Phi'(X^*)G(X^*) - [U_X]/[U_C]\}/[G'(X^*)]$

See Endress et al. (2014) for discussion and derivation of the extended Hotelling rule for a renewable resource in a model with production and amenity value.

4.3 *Impact of natural disaster on the renewable resource sector*

In the no-disaster scenario, capital accumulation and natural resource harvesting take their optimal paths when guided by the Ramsey and Hotelling rules, approaching steady state stock levels K^* and X^*. But suppose that a natural disaster occurs at time τ_1. We consider several possibilities involving direct impacts on the natural resource system.

4.3.1 Disaster does not push the system below the critical value tipping point

Suppose that a portion D_X of the resource stock is damaged at time τ_1, leaving $(1 - D_X)X(\tau_1) > X_{cr}$ remaining. Utility $U(C, X)$ takes a discontinuous drop at time τ_1 and then increases, much like the upper graph of Figure 1. Intertemporal welfare, $V(t)$, stays continuous and non-decreasing as in Figure 1, but is not differentiable at τ_1. In the long run, as time t approaches infinity, the system approaches the steady state levels K^* and X^*, which have not been affected by the natural disaster.

4.3.2 Disaster pushes the system below the critical value tipping point

Now suppose that a portion D_X of the resource stock is damaged at time τ_1, leaving $(1 - D_X)X(\tau_1) < X_{cr}$ remaining. In this case, the natural resource system collapses toward extinction; if the natural resource is an essential factor of production, and no substitutes exist, the economy fails and sustainability is not possible.

4.3.3 Disaster induces a reduction in the intrinsic growth rate

Another possibility is that the intrinsic growth rate, rather than the capital stock itself, is affected by the disaster. Figure 2, depicts two graphs for $G(X)$ with $\pi_1 > \pi_2$. The effect of a reduced intrinsic growth rate on the steady state stock X^* and harvest rate H^* is not immediately apparent; the signs of comparative statics $d[X^*]/d\pi$ and $d[H^*]/d\pi$ are heavily dependent on the specific functional forms for F, Φ, G, and U, as the extended Hotelling rule suggests. Farmer and Bednar-Friedl (2010) construct a case for which $d[X^*]/d\pi > 0$ and $d[H^*]/d\pi > 0$. In this case, a negative shock to the intrinsic growth rate, engendered by the natural disaster, lowers steady state values of the resource stock and the harvest rate. The effect would be to lower the golden rule level of utility $U(C^*, X^*)$.

5 Conclusions

In this preliminary integration of disaster economics and sustainability theory, we reach the following conclusions: In the long run, optimal paths of consumption and investment, whatever the source of uncertainty, lead to steady states defined by fundamentals of the economy: the production function, the utility function, and the natural resource growth function. In particular, neither the long run steady state, nor the speed of convergence to the steady state depends on produced capital or natural resource stocks in the initial period.

Short of a catastrophe that depresses natural capital below the tipping point,[10] a natural disaster functions as a restart, with new levels of capital starting from the time of disaster. In the case of recurrent, non-catastrophic disasters, the result is a series of restarts. Despite destruction of capital in the short run, economic recovery can be achieved in the medium run, and the economy can approach a golden rule steady state if optimal paths are followed. These possibilities require precautionary investment in productive and natural capital. Equally important for the longer run, they imply the avoidance of additional objectives such as economic-ecological self-sufficiency.

The golden rule, optimal steady state is also consistent with sustainability. In particular, the Arrow et al. (2004) sustainability criterion that intertemporal welfare not decline is satisfied by the optimal program. A sustainability constraint requiring consumption, utility, or intertemporal welfare to be non-declining would be redundant except perhaps in the extreme

[10] For an analysis of optimal investment in the face of a tipping point, see e.g. van der Ploeg and de Zeeuw (2016).

case of potential catastrophic, ecological collapse. In this case, optimality may be at odds with a sustainability constraint.

Both the discrete two-period model and the extended-Ramsey model deliver policy implications relating investment to probability of disaster. In the simple two-period model with capital at risk of partial destruction, an increase in the probability of disaster increases optimal investment in capital at risk for the "typical" case that the coefficient of relative risk aversion (preference for smoothing) is greater than one and decreases investment for $\eta < 1$. For the intermediate case of $\eta = 1$, the preference for consumption smoothing exactly offsets the tendency to decrease exposure of capital at risk such that optimal investment is invariant to the probability of disaster. For the continuous time model, the situation is more nuanced. In the typical case with $\eta > 1$, an increase in hazard rate P shifts the time path of optimal precautionary investment toward the present: higher precautionary investment earlier, and lower investment later.

We close with some suggested extensions to the model as opportunities for future research. One extension would be to include capital adjustment costs and capital hardening into these models. Relatedly, production and utility could be modeled more generally by including labor as arguments in the production and utility functions. Such a model would allow one to investigate how labor affects the optimal allocation between production and investment, as well as determine how this allocation varies with changes to the output elasticity of labor and the weight assigned to labor disutility.

A key pillar of sustainable growth is adopting a complex systems approach to modeling and analysis, integrating natural resource systems, the environment, and the economy. Much interdisciplinary research, both theoretical and empirical, remains to be done, and this is where the emerging field of sustainability science (Clark, 2007) has the potential to make a substantial contribution. For example, more sophisticated analysis of substitutability among different varieties of capital, especially in coupled systems, appears possible. Notions of network effects, emergence, spontaneous order, non-linearities, critical transitions, bifurcations, regime shifts, tipping points, and chaos in adaptive, complex systems, though regarded as "faddish" in some circles, are now being incorporated into advanced models. Scheffer (2009) provides an accessible overview of the issues and challenges involved.

In the big picture, research in this area will likely involve continued development and refinement of models linking sustainability theory, uncertainty, and the economics of natural disaster – e.g. including technological change, a central interest of endogenous growth theory – and disaster shocks to productivity, the province of business cycle theory. Further research in the realm of stochastic analysis is also required to construct better estimates of *P* and *D*. Though not considered a feature of the present two-period model, this line of research remains essential to inform policy for disaster risk management.

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Appendix

I Two-period model preliminaries

In our simple economy, production takes the form $F(K) = A(K)^{\alpha}$, with $0 < \alpha \leq 1$. The letter A designates productivity or the state of technology in production and α is the output elasticity of capital. We do not address technological change or negative productivity shocks, and labor is held constant. Investment at time 0 can be specified generally as $I = K_t - (1 - \delta)K_0$, where $0 \leq \delta \leq 1$ is the rate of capital depreciation each period.[11] For computational tractability in the two-period model, we assume 100% capital depreciation by taking $\delta = 1$ (e.g., as in Benassy, 2011; Romer, 2012).[12] Accordingly, K_t is period 0 investment in productive capital. The amount of capital that actually remains in the second period, however, is not necessarily equal to K_t. There is a probability P, that a natural disaster destroys a fraction (D) of the capital stock in period t. In other words, there are two possible end states: $K_{t,1} = (1 - D)K_t$ with probability P and $K_{t,2} = K_t$ with probability $(1 - P)$.

Since the uncertain capital stock is an input to production, period-t consumption is state dependent. Following the notation used for capital, $C_{t,1}$ denotes period t consumption in the

[11] We assume that consumption goods and capital goods are instantaneously and costlessly convertible. Examples ("stories") offered to justify this conventional convenience include rice (Gollier, 2013), corn (Acemoglu, 2008), and cattle ("breed 'em or eat 'em") economies. Alternatively, Pindyck and Wang (2013) combine installation/adjustment costs with Brownian motion models of capital diffusion and shocks. A more classic approach goes back to vintage putty-clay models of capital (Wan, 1971).

[12] In section 3, we move to continuous time and reintroduce depreciation more generally with $0 \leq \delta \leq 1$.

disaster scenario, while $C_{t,2}$ is consumption in the absence of disaster. The general specification for utility is $U(C) = [C^{(1-\eta)}]/(1 - \eta)$. The parameter η, with $0 \leq \eta$, is the absolute value of the consumption elasticity of marginal utility (i.e. $C[U''(C)]/[U'(C)] = -\eta$) and measures the preference for consumption smoothing. The parameter η also serves as the coefficient of relative risk aversion (to intertemporal inequality of consumption) and is positive for concave utility functions ($U''(C) < 0$). The case $\eta = 0$ corresponds to risk neutrality, and the case $\eta = 1$ is equivalent to the functional form $U(C) = \log C$.[13]

Expected utility over the two periods, assuming temporal separability, is given by $V = U(C_0) + \beta E[U(C_t)]$, where $\beta = e^{-\rho t}$ is the discount factor and $E[U(C_t)] = P[U(C_{t,1})] + (1 - P)[U(C_{t,2})]$. Note, that unless agents are risk neutral, $E[U(C_t)] \neq U[E(C_t)]$. The planner's objective is to maximize expected utility, subject to feasibility constraints in each of the two periods. Assuming that agents consume all of their total output in the second period and leave no terminal capital stock, the optimization problem can be stated as follows:

(A1) Max $V = U(C_0) + \beta E[U(C_t)]$, subject to

$$C_0 = F(K_0) - K_t = A(K_0)^\alpha - K_t$$

$$C_{t,1} = F(K_{t,1}) = F((1 - D)K_t) = A[(1 - D)K_t]^\alpha$$

$$C_{t,2} = F(K_{t,2}) = F(K_t) = A(K_t)^\alpha$$

Using the method of substitution, we maximize V over K_t:[14]

(A2) Max V, $V = \{U(A(K_0)^\alpha - K_t) + \beta[PU(A[(1 - D)K_t]^\alpha) + (1 - P)U(A(K_t)^\alpha)]\}$

The first order condition for equation (A2) is

(A3) $$\partial V/\partial K_t = -U'(A(K_0)^\alpha - K_t) + \beta\{P[\alpha A(1 - D)^\alpha (K_t)^{\alpha-1}]U'(A[(1 - D)K_t]^\alpha) + (1 - P)[\alpha A(K_t)^{\alpha-1}]U'(A(K_t)^\alpha)\} = 0$$

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[13] For $U(C) = \log C$, $U'(C) = C^{-1}$ and $U''(C) = -C^{-2}$, which together imply that $\eta = 1$.

[14] This method is equivalent to setting up a Lagrangian, but is simpler in this situation.

For the functional form $U(C) = [C^{(1-\eta)}]/(1 - \eta)$, the first order condition for expected utility maximization becomes

(A4) $(A(K_0)^{\alpha} - K_t)^{-\eta} = \beta\alpha(A)^{1-\eta}(K_t)^{\alpha(1-\eta)-1}\{P[(1 - D)^{\alpha(1-\eta)} - 1] + 1\}$, or

$$\{[A(K_0)^{\alpha} - K_t]^{-\eta}\}/\beta\{\{[A(K_t)^{\alpha}]^{-\eta}\}\{P[(1 - D)^{\alpha(1-\eta)} - 1] + 1\}\} = [\alpha A(K_t)^{\alpha-1}]$$

This is a marginal rate of substitution (MRS) = marginal rate of transformation (MRT) condition in the expected utility framework. The left hand side is $U'(C_0)/\beta E[U'(C_t)]$, while the right hand side $[\alpha A(K_t)^{\alpha-1}]$ = MRT = $-d(C_t)/d(C_0)$, which can be derived from the feasibility constraints. Note that equation (A4) is an *ex ante* formulation to guide investment. The *ex post* expression for MRT is more complicated and must be derived from the production possibility frontier for $U(C_0)$ and $EU(C_t)$.

II Two-period model comparative statics

We consider the effect of an increase in probability of disaster, P, on optimum investment in productive capital, K_t. That is, we derive and attempt to sign the comparative static $d(K_t)/dP$. First, we rewrite the first order condition (A4) as

(A5) $\{[A(K_0)^{\alpha} - K_t]^{-\eta}\}/\beta\{\alpha(A)^{1-\eta}(K_t)^{\alpha(1-\eta)-1}\} = \{P[(1 - D)^{\alpha(1-\eta)} - 1] + 1\}$

Taking natural logs of both sides of equation (A5) and using the approximation $\log(x + 1) \cong x$ on the right hand side yields

(A6) $-\eta\log[A(K_0)^{\alpha} - K_t] - \log[\beta\alpha] - (1- \eta)\log[A] - [\alpha(1- \eta) - 1]\log(K_t)$

$$\cong P[(1 - D)^{\alpha(1-\eta)} - 1]$$

Now we totally differentiate with respect to K_2 and P. After algebraic manipulation, this yields

(A7) $\{\eta/[A(K_0)^{\alpha} - K_t] - [\alpha(1 - \eta) - 1]/(K_t)\}d(K_t) \cong [(1 - D)^{\alpha(1-\eta)} - 1]dP$

Observe that the expression in curly brackets preceding the differential $d(K_t)$ on the left hand side of equation (A7) is strictly positive for $\alpha \leq 1$ and any value of η. For the typical case of $\eta > 1$,[15] it must be that $d(K_t)/dP > 0$, since $[(1 - D)^{\alpha(1-\eta)} - 1] > 0$. An increase in the probability of disaster induces greater investment in productive capital to support greater consumption in period t. In the case of $\eta = 1$, as considered e.g. by Stern (2007) and Nordhaus (2008), $[(1 - D)^{\alpha(1-\eta)} - 1] = 0$ such that $d(K_t)/dP = 0$. That is the preference for smoothing is just enough to exactly offset the avoidance of exposing capital to possible damage. For $\eta < 1$, $[(1 - D)^{\alpha(1-\eta)} - 1] < 0$, which means that $d(K_t)/dP < 0$. A higher probability of disaster induces less investment in capital to allow for greater consumption today. In summary, investment will decrease, remain unchanged or increase in response to an increase in the probability of disaster, depending on whether η is less than, equal to, or greater than 1.

III The effect of capital depreciation on precautionary investment

The extended Ramsey condition presented in section 2.2 was derived for the case of 100% capital depreciation ($\delta = 1$): $r_t = \rho + (1/t)\eta\log\{[A(K_t)^\alpha]/[A(K_0)^\alpha - K_t]\} - (1/t)\{P[(1 - D)^{\alpha(1-\eta)} - 1]\}$. How would a lower rate of capital depreciation, ($\delta < 1$), affect precautionary investment? Using several approximations, we estimate that for the case $1 < \eta$, a δ less than 1 tends to raise the level of precautionary investment at each time t; more capital is at risk. However, we conclude that the effect is negligible, especially in the continuous time model.

Expected utility across the two periods is $V = [U(C_0) + \beta EU(C_t)]$ and $E[U(C_t)] = P[U(C_{t,1})] + (1 - P)[U(C_{t,2})]$. The functional forms are $F(K) = A(K)^\alpha$, with $0 < \alpha \leq 1$ and $U(C) = [C^{(1-\eta)}]/(1 - \eta)$, where we assume $0 \leq \eta$.

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[15] Arrow (1999) and Dasgupta (2001) separately opine that η is 1.5. Dasgupta (2008) later argues that η should be between 2 and 4. Gollier (2013) advocates and Nordhaus (2008) and Weitzman (2007) consider the case of $\eta = 2$.

The planner's problem is

(A8) Max V, subject to:

$$C_0 = F(K_0) - K_t + (1 - \delta)K_0 = A(K_0)^{\alpha} - K_t + (1 - \delta)K_0$$

$$C_{t,1} = F(K_{t,1}) + (1 - \delta)(1 - D)K_t = A[(1 - D)K_t]^{\alpha} + (1 - \delta)(1 - D)K_t$$

$$C_{t,2} = F(K_{t,2}) + (1 - \delta)K_t = A(K_t)^{\alpha} + (1 - \delta)K_t$$

Maximizing over K_t yields the first order condition,

(A9) $$(A(K_0)^{\alpha} - K_t + (1 - \delta)K_0)^{-\eta} = \beta P\{\{A[(1 - D)K_t]^{\alpha} + (1 - \delta)(1 - D)K_t\}^{-\eta}\{\alpha A(1 - D)^{\alpha}(K_t)^{\alpha-1} + (1 - \delta)(1 - D)\}\} + \beta(1 - P)\{\{A(K_t)^{\alpha} + (1 - \delta)K_t\}^{-\eta}\{\alpha A(K_t)^{\alpha-1} + (1 - \delta)\}\}.$$

With extensive algebraic manipulation, the first order condition can be rearranged as

(A10) $$\{A(K_0)^{\alpha} - K_t + (1 - \delta)K_0\}^{-\eta}/\beta\{A(K_t)^{\alpha} + (1 - \delta)K_t\}^{-\eta} = \{P[(1 - D)^{\alpha(1-\eta)}(Z) - 1] + 1\}\{\alpha A(K_t)^{\alpha-1} + (1 - \delta)\},$$ where

(A11) $$Z = \{\{A(K_t)^{\alpha} + (1 - D)^{1-\alpha}(1 - \delta)K_t\}/\{A(K_t)^{\alpha} + (1 - \delta)K_t\}\}^{-\eta} *\{\{\alpha A(K_t)^{\alpha-1} + (1 - D)^{1-\alpha}(1 - \delta)\}/\{\alpha A(K_t)^{\alpha-1} + (1 - \delta)\}\}$$

$$= \{\{\alpha A(K_t)^{\alpha-1} + \alpha(1 - D)^{1-\alpha}(1 - \delta)\}/\{\alpha A(K_t)^{\alpha-1} + \alpha(1 - \delta)\}\}^{-\eta} *\{\{\alpha A(K_t)^{\alpha-1} + (1 - D)^{1-\alpha}(1 - \delta)\}/\{\alpha A(K_t)^{\alpha-1} + (1 - \delta)\}\}$$

Then assuming that α less than, but close to 1 in value,

(A12) $$Z \cong \{\{\alpha A(K_t)^{\alpha-1} + \alpha(1 - D)^{1-\alpha}(1 - \delta)\}/\{\alpha A(K_t)^{\alpha-1} + \alpha(1 - \delta)\}\}^{1-\eta} = \{\{MPK_t + \alpha(1 - D)^{1-\alpha}(1 - \delta)\}/\{MPK_t + \alpha(1 - \delta)\}\}^{1-\eta}$$

Now let $W = \{MPK_t + \alpha(1 - D)^{1-\alpha}(1 - \delta)\}/\{MPK_t + \alpha(1 - \delta)\}$. For $\alpha < 1$, we observe that $(1 - D)^{1-\alpha} < 1$ and so $W < 1$. Accordingly, for $1 < \eta$, $Z = \{W\}^{1-\eta} > 1$. When $\eta < 1$, $Z = \{W\}^{1-\eta} < 1$. And when $\eta =1$, $Z = 1$.

Finally, we consider the magnitude of the factor Z and its effect on precautionary investment as the economy evolves and productive capital is accumulated. In the early stages of capital accumulation, the marginal product of capital is large: MPK_t dominates $\alpha(1 - \delta)$. Hence, W and Z are close to 1, that the effect of Z on precautionary investment is small. As the economy grows, even in the face of natural disaster, the marginal product of capital declines and MPK_t no longer dominates in the expression for W. In fact, as the economy approaches the steady state, MPK_t approaches δ, and W approaches $\{\delta + \alpha(1 - D)^{1-\alpha}(1 - \delta)\}/\{\delta + \alpha(1 - \delta)\}$.

For illustration, we apply simple parameter values: $\alpha = 3/4$, $\delta = 1/4$, $\eta = 2$, and $D = 1/4$. These values yield $W = 0.8125$ and $Z = 1/ W = 1.2308$. While this appears to represent a large effect, it is significantly offset by the conditional probability density $f(t) = Pe^{-Pt}$. In continuous time, precautionary investment, adjusted for a rate of capital depreciation $0 \leq \delta < 1$, can be expressed as - $\{Pe^{-Pt}[(1 - D)^{\alpha(1-\eta)}(Z) - 1]\}$. For $1 < \eta$, the factor Z rises as time t increases, but density function weighting declines rapidly. Accordingly, we neglect the effect on precautionary investment of a capital depreciation rate less than 100%.